



TITLE:

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An obstruction to trivializing links by n -moves

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1 Introduction

The present article is a summary of our paper [8]. We refer the reader to [8] for more details and full proofs.

Let n be a positive integer. An n -move on a link is a local move as illustrated in Figure 1.1. Two links are n -move equivalent if they are transformed into each other by a finite sequence of n -moves. Note that if n is odd then n -moves may change the number of components of a link. Since a 2-move is generated by crossing changes and vice versa, we can consider an n -move as a generalization of a crossing change. Any link can be transformed into a trivial link by a finite sequence of crossing changes. Therefore, it is natural to ask whether or not any link is n -move equivalent to a trivial link. In 1980s, Yasutaka Nakanishi proved that all links with 10 or less crossings and Montesinos links are 3-move equivalent to trivial links, and he conjectured that any link is 3-move equivalent to a trivial link (see [7, Problem 1.59 (1)]). This conjecture is called the *Montesinos-Nakanishi 3-move conjecture*, and have been shown to be true for several classes of links, for example, all links with 12 or less crossings, closed 4-braids and 3-bridge links [1, 9, 11].

After 20 years, in [2, 3] M. K. Dąbkowski and J. H. Przytycki introduced the n th *Burnside group* of a link as an n -move equivalent invariant, and proved that for any odd



Figure 1.1: n -move

prime p there exist links which are not p -move equivalent to trivial links by using their p th Burnside groups. More precisely, they proved that the closure of the 5-braid $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$ and the 2-parallel of the Borromean rings are not 3-move equivalent to trivial links [2], and that the closure of the 3-braid $(\sigma_1\sigma_2)^6$ is not p -move equivalent to a trivial link for any prime number $p \geq 5$ [3]. That is, they gave counterexamples for the Montesinos-Nakanishi 3-move conjecture.

It is easy to see that the p th Burnside group is preserved by p -moves. While the p th Burnside group is a powerful invariant, it is hard to distinguish p th Burnside groups of given links in general. Hence to find a way to distinguish given Burnside groups is very important. In this article, we give an efficient way to distinguish p th Burnside groups of a given link and a trivial link (Theorem 3.1). In fact, by using Theorem 3.1, we show that there exist links, each of which is not p -move equivalent to a trivial link for any odd prime p (Theorem 3.3). Our method is naturally extended to both *virtual* and *welded* links. We prove that there exists a welded link which is not p -move equivalent to a trivial link for any odd prime p (Remark 3.5).

2 Burnside groups of links

Let L be a link in the 3-sphere S^3 and D an unoriented diagram of L . In [4, 5, 6, 13], a group $\Pi_D^{(2)}$ of D is defined as follows. Each arc of D yields a generator, and each crossing of D gives a relation $yx^{-1}yz^{-1}$, where x and z correspond to the underpasses and y corresponds to the overpass at the crossing, see Figure 2.1. The group $\Pi_D^{(2)}$ is an invariant of L . We call it the *associated core group* of L and denote it by $\Pi_L^{(2)}$.

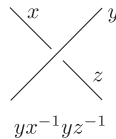


Figure 2.1: Relation of the associated core group

Remark 2.1. M. Wada [13] proved that $\Pi_L^{(2)}$ is isomorphic to the free product of the fundamental group of the double branched cover $M_L^{(2)}$ of S^3 branched along L and the infinite cyclic group \mathbb{Z} : $\Pi_L^{(2)} \cong \pi_1(M_L^{(2)}) * \mathbb{Z}$. Moreover, Dąbkowski and Przytycki [2, 3] pointed out that for a diagram D of L , $\pi_1(M_L^{(2)})$ is obtained from the group $\Pi_D^{(2)}$ of D by putting any fixed generator $x = 1$.

In [2, 3], for each positive integer n , Dąbkowski and Przytycki introduced n -move equivalence invariants of L by using $\Pi_L^{(2)}$ and $\pi_1(M_L^{(2)})$ as follows.

Definition 2.2 ([2, 3]). Suppose that $\Pi_L^{(2)} = \langle x_1, \dots, x_m \mid R \rangle$. Then $\pi_1(M_L^{(2)}) \cong \langle x_1, \dots, x_m \mid R, x_m \rangle$. Let W_n denote a set $\{w^n \mid w \in \langle x_1, \dots, x_m \rangle\}$, where $\langle x_1, \dots, x_m \rangle$ is the free group of rank m . The *unreduced n th Burnside group* $\widehat{B}_L(n)$ of L is defined as $\langle x_1, \dots, x_m \mid R, W_n \rangle$. The *n th Burnside group* $B_L(n)$ of L is defined as $\langle x_1, \dots, x_m \mid R, x_m, W_n \rangle$.

Proposition 2.3 ([2, 3]). $\widehat{B}_L(n)$ and $B_L(n)$ are preserved by n -moves.

We will focus on the unreduced n th Burnside group $\widehat{B}_L(n)$ from now on. Let $\widehat{B}_L^q(n)$ denote the quotient group of $\widehat{B}_L(n)$ by the q th term of the lower central series of $\widehat{B}_L(n)$ ($q = 1, 2, \dots$). We remark that $\widehat{B}_L(n)$ is not always finite but $\widehat{B}_L^q(n)$ is a finite group for all q , see for example [12, Chapter 2]. Then the proposition above immediately implies the following corollary.

Corollary 2.4. $\widehat{B}_L^q(n)$ and $|\widehat{B}_L^q(n)|$ are preserved by n -moves for any q .

Remark 2.5. Let \mathbb{Z}_n denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n . Let L be a link and D a diagram of L . A map $f : \{\text{arcs of } D\} \rightarrow \mathbb{Z}_n$ is a *Fox n -coloring* of D if f satisfies $f(x) + f(z) = 2f(y)$ for each crossing of D , where x and z correspond to the underpasses and y corresponds to the overpass at the crossing. The set of Fox n -colorings of D forms an abelian group and is an invariant of L . Moreover, it is known that the abelian group is isomorphic to $\widehat{B}_L^2(n)$ [10, Proposition 4.5].

3 Obstruction to trivializing links by p -moves

Let p be a prime number. The *Magnus \mathbb{Z}_p -expansion* E^p is a homomorphism from $\langle x_1, \dots, x_m \rangle$ into the formal power series ring in non-commutative variables X_1, \dots, X_m with \mathbb{Z}_p coefficients defined by $E^p(x_i) = 1 + X_i$ and $E^p(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \dots$ ($i = 1, \dots, m$). Then we have the following theorem.

Theorem 3.1 ([8, Theorem 4.1]). Let L be a link with $\Pi_L^{(2)} \cong \langle x_1, \dots, x_m \mid R \rangle$ and $\widehat{B}_L^2(p) \cong \mathbb{Z}_p^m$. If L is p -move equivalent to a trivial link, then for any $r \in R$,

$$E^p(r) = 1 + \sum_{(i_1, \dots, i_p)} c(i_1, \dots, i_p) X_{i_1} \cdots X_{i_p} + d(p+1)$$

for some $c(i_1, \dots, i_p) \in \mathbb{Z}_p$ such that $c(i_1, \dots, i_p) = c(i_{\sigma(1)}, \dots, i_{\sigma(p)})$ for any permutation σ of $\{1, \dots, p\}$, where (i_1, \dots, i_p) runs over $\{1, \dots, m\}^p$ and $d(k)$ denotes the terms of degree $\geq k$.

Even though 4 is not prime, we have the following theorem.

Theorem 3.2 ([8, Theorem 4.2]). *Let L be an m -component link with $\Pi_L^{(2)} \cong \langle x_1, \dots, x_m \mid R \rangle$. If L is 4-move equivalent to a trivial link, then for any $r \in R$,*

$$E^2(r) = 1 + \sum_{(i_1, i_2, i_3, i_4)} c(i_1, i_2, i_3, i_4) X_{i_1} X_{i_2} X_{i_3} X_{i_4} + d(5)$$

for some $c(i_1, i_2, i_3, i_4) \in \mathbb{Z}_2$ such that $c(i_1, i_2, i_3, i_4) = c(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}, i_{\sigma(4)})$ for any permutation σ of $\{1, 2, 3, 4\}$, where (i_1, i_2, i_3, i_4) runs over $\{1, \dots, m\}^4$.

By applying Theorem 3.1, we have the following theorem.

Theorem 3.3 ([8, Theorem 4.3]). *The closure of the 5-braid $(\sigma_1 \sigma_2 \sigma_3 \sigma_4)^{10}$ and the 2-parallel of the Borromean rings are not p -move equivalent to trivial links for any odd prime p .*

Remark 3.4. Dąbkowski and Przytycki proved Theorem 3.3 for $p = 3$ [2, Theorem 6]. In their proof, the condition that $p = 3$ is essential, and hence it seems hard to show Theorem 3.3 by using their arguments.

Proof of Theorem 3.3. Let γ be the 5-braid $(\sigma_1 \sigma_2 \sigma_3 \sigma_4)^{10}$ described by a diagram in Figure 3.1. We put labels x_i ($i = 1, 2, 3, 4, 5$) on initial arcs of the diagram. Progress from left to right, then the arcs are labeled by using relations of the associated core group. Thus we obtain labels Q_i of terminal arcs of γ as follows (see [2, Lemma 5]):

$$Q_i = x_1 x_2^{-1} x_3 x_4^{-1} x_5 x_1^{-1} x_2 x_3^{-1} x_4 x_5^{-1} x_i x_5^{-1} x_4 x_3^{-1} x_2 x_1^{-1} x_5 x_4^{-1} x_3 x_2^{-1} x_1.$$

Let $\bar{\gamma}$ be the closure of γ . Since we have relations $Q_i x_i^{-1}$ for $\Pi_{\bar{\gamma}}^{(2)}$, $\Pi_{\bar{\gamma}}^{(2)}$ has the presentation $\langle x_1, x_2, x_3, x_4, x_5 \mid r_1, r_2, r_3, r_4, r_5 \rangle$, where $r_i = Q_i x_i^{-1}$. We note that $\widehat{B}_{\bar{\gamma}}^2(p) \cong \mathbb{Z}_p^5$ for any odd prime p . On the other hand, by computing $E^p(r_1)$, then the coefficient of $X_2 X_3 X_4$ is 0 and that of $X_4 X_2 X_3$ is 2 in $E^p(r_1)$. Theorem 3.1 implies that $\bar{\gamma}$ is not p -move equivalent to a trivial link.

Let γ' be the 6-braid described by a diagram in Figure 3.2. We put labels x_i on initial arcs, y_i on terminal arcs, and Q_i on arcs of the diagram as illustrated in Figure 3.2 ($i = 1, 2, 3, 4, 5, 6$). By using relations of the associated core group, the labels Q_i are

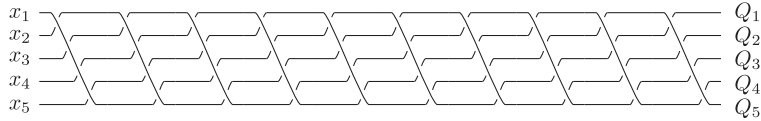


Figure 3.1: 5-braid $\gamma = (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^{10}$

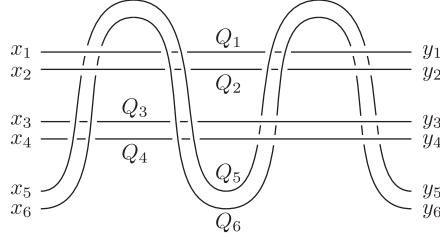


Figure 3.2: 6-braid γ' whose closure is the 2-parallel of the Borromean rings L_{2BR}

expressed as follows:

$$Q_i = \begin{cases} x_1 x_2^{-1} x_5 x_6^{-1} x_2 x_1^{-1} x_i x_1^{-1} x_2 x_6^{-1} x_5 x_2^{-1} x_1 \\ \quad = y_1 y_2^{-1} y_3 y_4^{-1} y_5 y_6^{-1} y_4 y_3^{-1} y_2 y_1^{-1} y_i y_1^{-1} y_2 y_3^{-1} y_4 y_6^{-1} y_5 y_4^{-1} y_3 y_2^{-1} y_1 & (i = 1, 2), \\ x_6 x_5^{-1} x_i x_5^{-1} x_6 \\ \quad = Q_6 Q_5^{-1} y_i Q_5^{-1} Q_6 = x_1 x_2^{-1} x_6 x_5^{-1} x_2 x_1^{-1} y_i x_1^{-1} x_2 x_5^{-1} x_6 x_2^{-1} x_1 & (i = 3, 4), \\ x_1 x_2^{-1} x_i x_2^{-1} x_1 = y_4 y_3^{-1} y_1 y_2^{-1} y_3 y_4^{-1} y_i y_4^{-1} y_3 y_2^{-1} y_1 y_3^{-1} y_4 & (i = 5, 6). \end{cases}$$

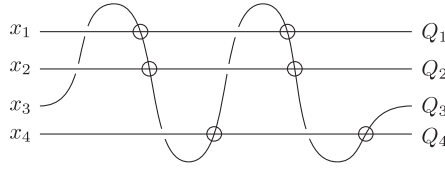
Since the closure of γ' is the 2-parallel of the Borromean rings L_{2BR} , $\Pi_{L_{2BR}}^{(2)}$ has the presentation $\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid r_1, r_2, r_3, r_4, r_5, r_6 \rangle$, where

$$r_i = \begin{cases} (x_1 x_2^{-1} x_5 x_6^{-1} x_2 x_1^{-1} x_i x_1^{-1} x_2 x_6^{-1} x_5 x_2^{-1} x_1)^{-1} \\ \quad \times x_1 x_2^{-1} x_3 x_4^{-1} x_5 x_6^{-1} x_4 x_3^{-1} x_2 x_1^{-1} x_i x_1^{-1} x_2 x_3^{-1} x_4 x_6^{-1} x_5 x_4^{-1} x_3 x_2^{-1} x_1 & (i = 1, 2), \\ (x_6 x_5^{-1} x_i x_5^{-1} x_6)^{-1} x_1 x_2^{-1} x_6 x_5^{-1} x_2 x_1^{-1} x_i x_1^{-1} x_2 x_5^{-1} x_6 x_2^{-1} x_1 & (i = 3, 4), \\ (x_1 x_2^{-1} x_i x_2^{-1} x_1)^{-1} x_4 x_3^{-1} x_1 x_2^{-1} x_3 x_4^{-1} x_i x_4^{-1} x_3 x_2^{-1} x_1 x_3^{-1} x_4 & (i = 5, 6). \end{cases}$$

We note that $\widehat{B}_{L_{2BR}}^2(p) \cong \mathbb{Z}_p^6$ for any odd prime p . On the other hand, by computing $E^p(r_6)$, then the coefficient of $X_2 X_4 X_6$ is 1 and that of $X_4 X_6 X_2$ is 0 in $E^p(r_6)$. Theorem 3.1 implies that L_{2BR} is not p -move equivalent to a trivial link. \square

Remark 3.5. For a welded link L , we can similarly define the *associated core group* $\Pi_L^{(2)}$ and the *unreduced n th Burnside group* $\widehat{B}_L(n)$ of L . We note that Theorems 3.1 and 3.2 hold for welded links. Hence, we can show that there exists a welded link which is not p -move equivalent to a trivial link for any odd prime p as follows. Let b be the welded 4-braid described by a virtual diagram in Figure 3.3. We put labels x_i and Q_i ($i = 1, 2, 3, 4$) on initial and terminal arcs of the diagram, respectively. By using relations of the associated core group, the labels Q_i are expressed as follows:

$$Q_i = \begin{cases} x_4 x_1^{-1} x_2 x_4^{-1} x_1 x_2^{-1} x_3 x_2^{-1} x_1 x_4^{-1} x_2 x_1^{-1} x_4 & \text{if } i = 3, \\ x_i & \text{otherwise.} \end{cases}$$

Figure 3.3: Welded 4-braid b

Let \bar{b} be the closure of b , then $\Pi_{\bar{b}}^{(2)} \cong \langle x_1, x_2, x_3, x_4 \mid Q_3 x_3^{-1} \rangle$. We note that $\widehat{B}_{\bar{b}}^2(p) \cong \mathbb{Z}_p^4$ for any odd prime p . On the other hand, by computing $E^p(Q_3 x_3^{-1})$, we have that the coefficient of $X_4 X_2 X_3$ is 1 and that of $X_4 X_3 X_2$ is 0 in $E^p(Q_3 x_3^{-1})$. Therefore, we have that \bar{b} is not p -move equivalent to a trivial link by Theorem 3.1.

Remark 3.6. All of the three links $\bar{\gamma}$, L_{2BR} and \bar{b} above are not 4-move equivalent to trivial links by Theorem 3.2 because terms of degree 3 survive in $E^2(r)$ for some relation r of $\Pi_L^{(2)}$ ($L = \bar{\gamma}, L_{2BR}, \bar{b}$).

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